Solution of reduced equations derived with singular perturbation methods

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For singular perturbation problems in dynamical systems, various appropriate singular perturbation methods have been proposed to eliminate secular terms appearing in the naive expansion. For example, the method of multiple time scales, the normal form method, center manifold theory, and the renormalization group method are well known. It is shown that all of the solutions of the reduced equations constructed with those methods are exactly equal to the sum of the most divergent secular terms appearing in the naive expansion. For the proof, a method to construct a perturbation solution which differs from the conventional one is presented, where we make use of the theory of the Lie symmetry group.

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I. INTRODUCTION

This paper investigates perturbation analysis of the fundamental system of nonlinear ordinary differential equations,

$$\frac{du}{dt} \coloneqq \dot{u} = Mu + \varepsilon f(u), \tag{1}$$

where $u \in \mathbb{C}^n$ is the dependent variable, M is a constant $n \times n$ matrix, $f: \mathbb{C}^n \to \mathbb{C}^n$ is a nonlinear function of u, and $\varepsilon \in \mathbb{R}$ is a perturbation parameter. In what follows, we refer to $\dot{u}=Mu$ as the unperturbed system and (1) as the perturbed system. The simplest perturbation solution is the naive expansion. Let us pose an expansion for the solution in powers of ε:

$$u(t;\varepsilon) = \sum_{p=0}^{\infty} \varepsilon^p u^{(p)}(t). \tag{2}$$

If we substitute Eq. (2) into Eq. (1), expand both sides of the equation with respect to ε, and equate the coefficients of each power of ε, then we obtain the following series of differential equations:

$$\dot{u}^{(0)} = Mu^{(0)},$$

$$\dot{u}^{(p)} = Mu^{(p)} + f^{(p-1)}(u^{(0)},u^{(1)}, \ldots ,u^{(p-1)}) \quad \text{for } p = 1,2, \ldots , \tag{3}$$

where

$$f\left(\sum_{p=0}^{\infty} \varepsilon^p u^{(p)}\right) \eqqcolon \sum_{p=0}^{\infty} \varepsilon^p f^{(p)}(u^{(0)},u^{(1)}, \ldots ,u^{(p)}). \tag{4}$$

If we solve these equations recursively, we find the naive expansion.

In this paper, we are especially interested in singular perturbation problems where secular terms arise in the naive expansion. In general, if $f(u)$ is a power series, secular terms arise in the naive expansion as we see later. To eliminate those secular terms, various appropriate methods are developed such as, for example, the renormalization group method [1–6], the method of multiple time scales [7], canonical Hamiltonian perturbation theory [8], averaging methods [9], the method of normal forms [7], center manifold theory [10], and so on. We refer to these methods simply as singular perturbation methods in this paper. It is well known that all of these methods result in equations all of which are equivalent that govern the long-time behavior of the system. Each of them is the dynamics for integral constants of an unperturbed system, or in other words, dynamics in the null space of the linear operator determined from the unperturbed system. Although the name of that equation such as the renormalization group equation or the normal form depends on the method, in this paper, we refer to it simply as the reduced equation.

There have been many studies which show that those singular perturbation methods surely lead to the well-behaving approximate solution. However, what the solution of that reduced equation exactly includes has not been clear. In this paper, we reveal the exact solution of the reduced equation. To be precise, the following statement is the main result shown in this paper.

Main result: the solution of the reduced equation up to first order for singular perturbation problem (1) is equal to sum of those terms which are proportional to $\varepsilon t, \varepsilon^2 t^2, \varepsilon^3 t^3, \ldots$ *in the naive expansion.*

In what follows, we refer to those secular terms as the most divergent terms in the naive expansion. Although this fact has been believed to be true in some cases [11], this is rigorously proved in this paper.

In the proof of the result, we first present another method to construct a perturbation solution—that is, Proposition 1 in Sec. II. While $f^{(p)}$ in Eqs. (3) must be more complicated function of $u^{(0)}, u^{(1)}, \ldots , u^{(p)}$ as p becomes large in general. The method presented in Sec. II leads to another recursive equation which has a clearer expression compared with Eqs. (3). In the derivation of those recursive equations, we make use of the Lie symmetry group which leaves the system, Eq. (1), invariant. This method can be interpreted as an extension of the renormalization group method with Lie symmetry group [6]. The recursive equation plays an important role in Sec. III for the proof of the main result.

II. METHOD TO CONSTRUCT A PERTURBATION SOLUTION

Consider the system of nonlinear ordinary differential equations as follows:

$$\frac{du}{dt} := \dot{u} = Mu + \varepsilon f(u), \tag{5}$$

where $u=u(t) \in \mathbb{C}^n$ is a vector-valued function of an independent valuable, M is an $n\times n$ matrix whose coefficients are constant, ε is a constant, and $f: \mathbb{C}^n \to \mathbb{C}^n$ is a smooth vector-valued function. In what follows, we refer to $\dot{u}=Mu$ as the unperturbed system and (5) as the perturbed system, and the solution of the system (5) is denoted by $u=u(t;\varepsilon)$ for the dependence to the perturbation parameter ε.

First, let us find a method to construct a perturbation solution. For the construction, we make use of the Lie symmetry method [12].

Proposition 1. Suppose $\psi(t,u;\varepsilon) \in \mathbb{C}^n$ is a vector-valued function of t, u, and ε and its formal expansion in powers of ε, $\psi(t,u;\varepsilon) =: \sum_{r=0}^{\infty} \varepsilon^r \psi^{(r)}(t,u)$, is admitted. Then, for $\psi(t,u;\varepsilon)$ which satisfies the recursive differential equations

$$L\psi^{(0)} = f,$$

$$L\psi^{(r)} = \psi^{(r-1)}\partial_u f - f\,\partial_u \psi^{(r-1)}, \quad \text{for } r=1,2,3,\ldots,$$

$$L := I[\partial_t + (Mu)\partial_u] - M, \tag{6}$$

the solution of system (5) satisfies

$$u(t;\varepsilon) = u(t;0) + \sum_{r=0}^{\infty} \int_0^{\varepsilon} \sigma^r \psi^{(r)}(t,u(t;\sigma))d\sigma. \tag{7}$$

Here I in the definition of L denotes the identity matrix.

Proof. Suppose Eq. (5) admits a Lie symmetry group whose infinitesimal generator is denoted by

$$X := \partial_\varepsilon + \psi(t,u;\varepsilon)\partial_u. \tag{8}$$

Then its prolongation X^*,

$$X^* = \partial_\varepsilon + \psi(t,u;\varepsilon)\partial_u + \psi^{\dot{u}}(t,u,\dot{u};\varepsilon)\partial_{\dot{u}},$$

$$\psi^{\dot{u}}(t,u,\dot{u};\varepsilon) := [\partial_t + \dot{u}\,\partial_u]\psi(t,u;\varepsilon), \tag{9}$$

satisfies the infinitesimal criterion of invariance of system (5)—that is,

$$X^*\,[\dot{u} - Mu - \varepsilon f(u)]|_{\text{Eq. (5)}} = 0. \tag{10}$$

Equation (10) reads

$$[I(\partial_t + (Mu)\partial_u)\psi - M\psi - f] + \varepsilon[f\,\partial_u\psi - \psi\,\partial_u f] = 0. \tag{11}$$

For the formal expansion in powers of ε,

$$\psi(t,u;\varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \psi^{(r)}(t,u), \tag{12}$$

by substituting Eq. (12) into Eq. (11) and equating the coefficient of each ε^r, we find recursive equations as follows:

$$L\psi^{(0)} = f, \tag{13}$$

$$L\psi^{(r)} = [\psi^{(r-1)}\partial_u f - f\,\partial_u \psi^{(r-1)}], \quad \text{for } r=1,2,\ldots,$$

$$L := I[\partial_t + (Mu)\partial_u] - M. \tag{14}$$

Solving Eqs. (13) and (14) recursively, we obtain a formal expansion of the infinitesimal generator of a Lie symmetry group which leaves the system (5) invariant. Then the solution of the system (5), $u=u(t;\varepsilon)$, invariant to X satisfies

$$X\,[u - u(t;\varepsilon)]|_{u=u(t;\varepsilon)} = 0. \tag{15}$$

Equation (15) reads

$$\frac{\partial}{\partial\varepsilon}u(t;\varepsilon) = \psi(t,u(t;\varepsilon);\varepsilon), \tag{16}$$

$$\Leftrightarrow u(t;\varepsilon) = u(t;0) + \sum_{r=0}^{\infty} \int_0^{\varepsilon} \sigma^r \psi^{(r)}(t,u(t;\sigma))d\sigma. \tag{17}$$

Thus, the integral equation (7) for solution of the system (5) has been obtained. ■

It should be remarked here that it is not necessary for ε to be small in this proposition. Therefore, Eq. (16) holds not only for perturbed systems, but also for generic systems which take the form of Eq. (5), although it seems to be practical for perturbation problems.

III. SOLUTION OF REDUCED EQUATIONS

Next, consider those systems whose linear part can be diagonalized. Then the system (5) reads

$$\dot{z} = \Lambda z + \varepsilon g(z), \quad z \in \mathbb{C}^n, \tag{18}$$

with a linear transformation from u into z. Here Λ is an $n\times n$ diagonal matrix whose components are denoted by $\Lambda_{ij} =: \delta_{ij}\lambda_i$, and g is the nonlinear vector-valued function constructed from f with the transformation. Then the recursive equations corresponding to Eqs. (6) become

$$L_i\phi_i^{(0)} = g_i,$$

$$L_i\phi_i^{(r)} = \sum_{j=1}^{n} [\phi_j^{(r-1)}\partial_{z_j} g_i - g_j \partial_{z_j}\phi_i^{(r-1)}], \quad \text{for } r=1,2,\ldots,$$

$$L_i := \left(\partial_t + \sum_{k=1}^{n} \lambda_k z_k \partial_{z_k}\right) - \lambda_i, \tag{19}$$

for a vector-valued function, $\phi(t,z;\varepsilon) \in \mathbb{C}^n$ for $r=0,1,\ldots$. Here and in what follows, the components of vectors and matrices are explicitly denoted for clarification of the following discussion, and equations hold for $i=1,\ldots,n$. In the same way we have derived Eq. (7), it follows that the solution of Eq. (18), $z=z(t;\varepsilon)$, satisfies

$$z_i(t;\varepsilon) = z_i(t;0) + \sum_{r=0}^{\infty} \int_0^{\varepsilon} \varepsilon^r \phi_i^{(r)}(t,z(t;\varepsilon))d\varepsilon. \tag{20}$$

As we see later, if we obtain $\{\phi^{(r)}(t,z)\}$, we can construct the naive expansion using Eq. (20) with the iterative method since we know the solution of the unperturbed system, $z(t;0)=e^{\Lambda t}z_0$ where z_0 is a constant.

Now we can show the following proposition.

Proposition 2. Suppose the nonlinear function in Eq. (18) is power series such as

$$g_i(z) = \sum_{p_1,p_2,\ldots,p_n=0}^{\infty} C^i_{p_1p_2\cdots p_n}\prod_{k=1}^{n} z_k^{p_k}, \tag{21}$$

where each $C^i_{p_1p_2\cdots p_n}$ is constant. Then there is a solution of Eqs. (19) which becomes power series of t and z which satisfies $\phi_i^{(r)}=O(t^r)$ for $r=1,2,\ldots$, while $\phi_i^{(0)}=O(t)$.

Proof. First, we seek $\phi^{(0)}$. According to Eqs. (19), it is the solution of the differential equation

$$L_i\phi_i^{(0)}(t,z) = \sum_{p_1,p_2,\ldots,p_n=0}^{\infty} C^i_{p_1p_2\cdots p_n}\prod_{k=1}^{n} z_k^{p_k}. \tag{22}$$

Note that, for arbitrary $(p_1,p_2,\ldots,p_n)\in\mathbb{N}^n$, $\Pi_{k=1}^n z_k^{p_k}$ are eigenfunctions of L_i. Among inhomogeneous terms on the right-hand side (rhs), those which satisfy the resonance condition $\Sigma_{j=1}^n\lambda_j p_j-\lambda_i=0$ cause secular terms in the solution. Then we obtain

$$\phi_i^{(0)}(t,z) = \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i=0}} C^i_{p_1p_2\cdots p_n} t\prod_{j=1}^{n} z_j^{p_j} + \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i\neq 0}} \frac{C^i_{p_1p_2\cdots p_n}}{\Sigma_{j=1}^n\lambda_j p_j-\lambda_i}\prod_{j=1}^{n} z_j^{p_j}. \tag{23}$$

Next, we seek $\phi^{(1)}$. According to Eqs. (19), it is the solution of the differential equation

$$L_i\phi_i^{(1)} = \sum_{j=0}^{n}[\phi_j^{(0)}\partial_{z_j}g_i - g_j\partial_{z_j}\phi_i^{(0)}]. \tag{24}$$

By virtue of $\phi^{(0)}$, the inhomogeneous terms in Eq. (24) can be split into four parts as

$$[\text{rhs of Eq. (24)}] = \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i=0}} E^i_{p_1p_2\cdots p_n} t\prod_{j=1}^{n} z_j^{p_j} + \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i\neq 0}} F^i_{p_1p_2\cdots p_n} t\prod_{j=1}^{n} z_j^{p_j} + \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i=0}} G^i_{p_1p_2\cdots p_n}\prod_{j=1}^{n} z_j^{p_j} + \sum_{\substack{p_1,p_2,\ldots,p_n \\ \Sigma_{j=1}^n\lambda_j p_j-\lambda_i\neq 0}} H^i_{p_1p_2\cdots p_n}\prod_{j=1}^{n} z_j^{p_j}, \tag{25}$$

for some constants $\{E^i_{p_1p_2\cdots p_n},F^i_{p_1p_2\cdots p_n},G^i_{p_1p_2\cdots p_n},H^i_{p_1p_2\cdots p_n}\}$. All terms in the first part seem to cause secular terms in $\phi^{(1)}$ which are proportional to t^2 since each of them satisfies resonance condition. However, we can show that the first part vanishes by substituting Eqs. (21) and (23) into the right-hand side of Eq. (24) and calculating $\{E^i_{p_1p_2\cdots p_n}\}$. The calculation is found in Appendix concretely. Therefore, the most divergent terms in $\phi^{(1)}$ are not proportional to t^2, but proportional to t. For $r=2,3,\ldots$, inhomogeneous terms in Eq. (19) which are proportional to t^{r-1} and which satisfy the resonance condition remain in general. Then those inhomogeneous terms cause secular terms proportional to t^r in $\phi^{(r)}$. ■

Now we can find the solution of reduced equations which result from various singular perturbation methods.

Corollary. For the system of differential equations,

$$\frac{\partial z(t;\varepsilon)}{\partial\varepsilon} = t\phi_{sec}^{(0)}(z(t;\varepsilon)), \tag{26}$$

with $z(t;0)$ denoting the solution of the unperturbed system, the solution is equal to sum of terms proportional to $\varepsilon t,\varepsilon^2t^2,\ldots,\varepsilon^n t^n,\ldots$ in the naive expansion of the system (18). Here $\phi^{(0)}$ is split into $\phi^{(0)}(t,z)=:t\phi_{sec}^{(0)}(z)+\phi_{non}^{(0)}(z)$ by virtue of Eq. (23).

Proof. According to Eq. (20),

$$z(t;\varepsilon) = z(t;0) + \int_0^{\varepsilon} t\phi_{sec}^{(0)}(z(t;\sigma))d\sigma + \int_0^{\varepsilon}\phi_{non}^{(0)}(z(t;\sigma))d\sigma + \int_0^{\varepsilon}\varepsilon\phi^{(1)}(t,z(t;\sigma))d\sigma + \int_0^{\varepsilon}\varepsilon^2\phi^{(2)}(t,z(t;\sigma))d\sigma + \cdots. \tag{27}$$

Thanks to this self-consistent integral equation, we can construct the naive expansion with the iterative method. In terms of Proposition 2, it follows that terms proportional to $\varepsilon t,\varepsilon^2t^2,\ldots,\varepsilon^n t^n,\ldots$ in the naive expansion arise only from the term $\int_0^{\varepsilon}t\phi_{sec}^{(0)}(t;\sigma)d\sigma$ among terms on the right-hand side of Eq. (27) any step of the iteration. Therefore, the solution of the following equation (29) is exactly equal to the sum of terms proportional to $\varepsilon t,\varepsilon^2t^2,\ldots,\varepsilon^n t^n,\ldots$ in the naive expansion:

$$z(t;\varepsilon) = z(t;0) + \int_0^{\varepsilon} t\phi_{sec}^{(0)}(z(t;\sigma))d\sigma, \tag{28}$$

$$\Leftrightarrow\frac{\partial z(t;\varepsilon)}{\partial\varepsilon} = t\phi_{sec}^{(0)}(z(t;\varepsilon)), \tag{29}$$

where we adopt the solution of the unperturbed system as $z(t,0)$. ■

To complete the proof of the main result, we have to show various the widely accepted reduced equations is equivalent to Eq. (26). As a result of singular perturbation methods, we obtain reduced equations such as

$$\frac{\partial z(t;\varepsilon)}{\partial t} = \varepsilon z_{sec}^{(1)}(z(t;\varepsilon)), \tag{30}$$

where $z^{(1)}$ denotes the coefficient of ε in the naive expansion and we set $z^{(1)}(t,z^{(0)}) =: tz_{sec}^{(1)}(z^{(0)}) + z_{non}^{(1)}(z^{(0)})$. Equation (30) is a normal-form expression of the reduced equations. Although the well-known normal form contains a linear part [7] such as

$$\frac{\partial \tilde{z}(t;\varepsilon)}{\partial t} = \Lambda\tilde{z}(t;\varepsilon) + \varepsilon\phi_{sec}^{(0)}(\tilde{z}(t;\varepsilon)), \tag{31}$$

Eq. (31) reads Eq. (32) under $z := \exp(-\Lambda t)\tilde{z}$. We can transform Eq. (32) into a renormalization group equation or equivalent reduced equations derived with other methods if we adopt integral constants appearing in the solution of the unperturbed system as dependent variables [1]. The equivalence of the normal-form theory and the renormalization group method is discussed in [13] in detail.

Equation (30) reads

$$\frac{\partial z(t;\varepsilon)}{\partial t} = \varepsilon\phi_{sec}^{(0)}(z(t;\varepsilon)) \tag{32}$$

for the following reason: For the expanded form of the solution, $z(t;\varepsilon) =: \sum_{k=0}^{\infty}\varepsilon^k z^{(k)}(t)$, $z^{(1)}$ satisfies

$$z_i^{(1)}(t) = \lambda_i z_i^{(1)}(t) + g_i(z^{(0)}(t)) \Leftrightarrow \left(\partial_t + \sum_{k=1}^{n} z_k^{(0)}\partial_{z_k^{(0)}}\right) z_i^{(1)}(t,z^{(0)})$$
$$= \lambda_i z_i^{(1)}(t,z^{(0)}) + g_i(z^{(0)}) \Leftrightarrow \left(\partial_t + \sum_{k=1}^{n}\lambda_k z_k^{(0)}\partial_{z_k^{(0)}} - \lambda_i\right) z_i^{(1)}$$
$$\times(t,z^{(0)}) = g_i(z^{(0)}). \tag{33}$$

Equation (33) corresponds to the first equation of (19) if we replace $z^{(1)}$ and $z^{(0)}$ with $\phi^{(0)}$ and z, respectively. With a new independent valuable $\tau := \varepsilon t$, both Eqs. (26) and (32) can be written as

$$\frac{dz(\tau)}{d\tau} = \phi_{sec}^{(0)}(z(\tau)). \tag{34}$$

Thus, we have shown that the solution of the reduced equations is equal to the sum of the most divergent terms in the naive expansion when we construct the reduced equations up to only first order.

IV. EXAMPLE: THE DUFFING EQUATION

Let us see what is shown above holds through a simple example. Consider the Duffing equation

$$\ddot{u} + u = \varepsilon u^3. \tag{35}$$

Introducing $z := u + i\dot{u}$ for simplicity, we have

$$\dot{z} + iz = \varepsilon\frac{i}{8}(z+\bar{z})^3. \tag{36}$$

First, let us review the proof of the main result with this example. Suppose Eq. (36) admits the Lie symmetry group whose infinitesimal generator is denoted by

$$X := \partial_\varepsilon + \psi^z(t,z,\bar{z};\varepsilon)\partial_z + \psi^{\bar{z}}(t,z,\bar{z};\varepsilon)\partial_{\bar{z}}. \tag{37}$$

Note that it can be shown $\psi^{\bar{z}}(t,z,\bar{z};\varepsilon) = \overline{\psi^{z}(t,z,\bar{z};\varepsilon)}$. Then its prolongation X^*,

$$X^* = \partial_\varepsilon + \psi^z(t,z,\bar{z};\varepsilon)\partial_z + \psi^{\bar{z}}(t,z,\bar{z};\varepsilon)\partial_{\bar{z}} + \psi^{\dot{z}}(t,z,\bar{z},\dot{z},\dot{\bar{z}};\varepsilon)\partial_{\dot{z}}$$
$$+ \psi^{\dot{\bar{z}}}(t,z,\bar{z},\dot{z},\dot{\bar{z}};\varepsilon)\partial_{\dot{\bar{z}}},$$

$$\psi^{\dot{z}}(t,z,\bar{z},\dot{z},\dot{\bar{z}};\varepsilon) := [\partial_t + \dot{z}\partial_z + \dot{\bar{z}}\partial_{\bar{z}}]\psi^z(t,z,\bar{z};\varepsilon),$$
$$= \overline{\psi^{\dot{\bar{z}}}(t,z,\bar{z},\dot{z},\dot{\bar{z}};\varepsilon)}, \tag{38}$$

satisfies the infinitesimal criterion of the invariance corresponding to Eq (10):

$$X^*\left[\dot{z} + iz - \varepsilon\frac{i}{8}(z+\bar{z})^3\right]\Bigg|_{\text{Eq. (36)}} = 0. \tag{39}$$

For the formal expanded form of ψ,

$$\psi^z(t,z,\bar{z};\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r\psi^{(r)}(t,z,\bar{z}), \tag{40}$$

the equation for the leading order becomes

$$(\partial_t - iz\partial_z + i\bar{z}\partial_{\bar{z}} + i)\psi^{(0)}(t,z,\bar{z}) = \frac{i}{8}(z+\bar{z})^3. \tag{41}$$

Solving this, we obtain

$$\psi^{(0)}(t,z,\bar{z}) = -\frac{1}{16}z^3 + \frac{3i}{8}t|z|^2 z + \frac{3}{16}|z|\bar{z} + \frac{1}{32}\bar{z}^3. \tag{42}$$

The reduced equation corresponding to Eq. (26) becomes

$$\frac{\partial z(t,\varepsilon)}{\partial\varepsilon} = \frac{3i}{8}t|z(t;\varepsilon)|^2 z(t;\varepsilon). \tag{43}$$

With the integral equation expression corresponding to (28), we can find the solution with iterative method. The solution up to third order becomes

$$z(t,\varepsilon) = Ae^{-it} + \frac{3i}{8}\varepsilon t|A|^2Ae^{-it} - \frac{9}{128}\varepsilon^2t^2|A|^4Ae^{-it}$$
$$- \frac{9i}{1024}\varepsilon^3t^3|A|^6Ae^{-it} + \cdots, \tag{44}$$

where A denotes the integral constant. We can immediately show this solution is exactly equal to the most divergent terms in the naive expansion by constructing it directly.

Next, let us see Eq. (43) is equivalent to reduced equations derived with conventional singular perturbation methods. Although there are many ways to represent the reduced equations, one of them is the normal form [7]

$$\frac{d\tilde{z}(t)}{dt} = -i\tilde{z}(t) + \varepsilon\frac{3i}{8}|\tilde{z}(t)|^2\tilde{z}(t), \tag{45}$$

which corresponds to (31). Under $\tilde{z} =: A(t)e^{-it}$, Eq. (45) reads

$$\frac{dA(t)}{dt} = \varepsilon \frac{3\mathrm{i}}{8}|A(t)|^2 A(t), \quad (46)$$

which corresponds to (30). If we set $A(t) =: R(t)e^{-\mathrm{i}\theta(t)}$ where $R(t),\ \theta(t) \in \mathbb{R}$, the reduced equation reads

$$\frac{dR}{dt} = 0, \quad (47)$$

$$\frac{d\theta}{dt} = -\varepsilon \frac{3}{8} R^2, \quad (48)$$

which is called renormalization group equation [1,2]. Under $\tau := \varepsilon t$, both Eq. (43) and Eq. (46) read

$$\frac{d\hat{z}(\tau)}{d\tau} = \frac{3\mathrm{i}}{8}|\hat{z}(\tau)|^2 \hat{z}(\tau). \quad (49)$$

Thus, equivalence has been shown for the Duffing equation.

V. CONCLUDING REMARKS

The main purpose of this paper has been the derivation of an exact solution of the reduced equations which result from singular perturbation methods. What has been shown is that the solution of the reduced equations up to first order is equal to the sum of the most divergent terms, which are proportional to $\varepsilon t, \varepsilon^2 t^2, \varepsilon^3 t^3, \ldots$ appearing in the naive expansion. In other words, taking up to only first order with respect to perturbation parameter is enough to include those most divergent terms in the approximate solution. The main result has been proved without any approximation. Then it holds not only in the case where ε is small, although this result is meaningful in the context of perturbation analysis.

Another result has been presented in this paper. That is a method to construct a perturbation solution where we make use of the Lie symmetry group which leaves the system invariant. With this method, we obtain recursive equations (6) instead of Eqs. (3).

For the future, it should be investigated how the approximation improves if higher-order terms are taken into consideration when we construct the reduced equation. Another interest is the application to systems of partial differential equations (PDEs). In some PDE systems, it has been shown that, in constructing reduced equations, we should take up not only most divergent terms in the naive expansion, but also other terms to preserve the symmetry of the original system [14]. Therefore, the proof presented in this paper should be modified properly to those systems.

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APPENDIX: THE CALCULATION OF $E^i_{p_1 p_2 \cdots p_n}$ IN EQ. (25)

At first, we consider the first term on the right-hand side of Eq. (24), $\phi_j^{(0)}\partial_{z_j} g^i$. Substituting Eqs. (21) and (23) into Eq. (24) and writing terms proportional to t, we obtain

$$\phi_j^{(0)}\partial_{z_j} g^i = \left(\sum_{\substack{p_1,p_2,\ldots,p_n \\ \sum_{k=1}^n \lambda_k p_k - \lambda_j = 0}} C^j_{p_1 p_2 \cdots p_n} t \prod_{k=1}^n z_k^{p_k} \right) \times \partial_{z_j}\left(\sum_{q_1,q_2,\ldots,q_n} C^i_{q_1 q_2 \cdots q_n} \prod_{k=1}^n z_k^{q_k} \right) + [\text{terms not proportional to } t]. \quad (A1)$$

Those terms which are zero eigenfunctions of $\sum_{k=0}^n \lambda_k z_k \partial_{z_k} - \lambda_i$ cause terms proportional to t^2 in $\phi_i^{(1)}$. Such terms on the right-hand side satisfy the resonance condition

$$\sum_{\substack{k=1 \\ k\neq j}}^n \lambda_k (p_k + q_k) + \lambda_j [p_j + (q_j - 1)] - \lambda_i = 0, \quad (A2)$$

which reads

$$\sum_{k=1}^n \lambda_k q_k - \lambda_i = 0, \quad (A3)$$

since $\sum_{k=1}^n \lambda_k p_k - \lambda_j = 0$. Then, the resonant terms which are proportional to t on the right-hand side of Eq. (A1) become

$$t\left(\sum_{\substack{p_1,p_2,\ldots,p_n \\ \sum_{k=1}^n \lambda_k p_k - \lambda_j = 0}} C^j_{p_1 p_2 \cdots p_n} \prod_{k=1}^n z_k^{p_k} \right) \times \partial_{z_j}\left(\sum_{\substack{q_1,q_2,\ldots,q_n \\ \sum_{k=1}^n \lambda_k q_k - \lambda_i = 0}} C^i_{q_1 q_2 \cdots q_n} \prod_{k=1}^n z_k^{q_k} \right). \quad (A4)$$

On the other hand, for the second term on the right-hand side of Eq. (25), $g^j \partial_{z_j} \phi_i^{(0)}$,

$$g_j \partial_{z_j} \eta_i^{(0)} = \left(\sum_{q_1,q_2,\ldots,q_n} C^j_{q_1 q_2 \cdots q_n} \prod_{k=1}^n z_k^{q_k} \right) \times \partial_{z_j}\left(\sum_{\substack{p_1,p_2,\ldots,p_n \\ \sum_{k=1}^n \lambda_k p_k - \lambda_i = 0}} C^i_{p_1 p_2 \cdots p_n} t \prod_{k=1}^n z_k^{p_k} \right) + [\text{terms not proportional to } t]. \quad (A5)$$

The resonance condition in this case becomes

$$\sum_{\substack{k=1\\k\neq j}}^{n} \lambda_k(p_k+q_k)+\lambda_j[(p_j-1)+q_j]-\lambda_i=0, \tag{A6}$$

which reads

$$\sum_{k=1}^{n} \lambda_k q_k-\lambda_j=0, \tag{A7}$$

since $\Sigma_{k=1}^{n}\lambda_k p_k-\lambda_i=0$. Then, the resonant terms which are proportional to t on the right-hand side of Eq. (A5) become

$$t\left(\sum_{\substack{q_1,q_2,\ldots,q_n\\ \Sigma_{k=1}^{n}\lambda_k q_k-\lambda_j=0}} C^j_{q_1q_2\cdots q_n}\prod_{k=1}^{n} z_k^{q_k}\right)\times\partial_{z_j}\left(\sum_{\substack{p_1,p_2,\ldots,p_n\\ \Sigma_{k=1}^{n}\lambda_k p_k-\lambda_i=0}} C^i_{p_1p_2\cdots p_n}\prod_{k=1}^{n} z_k^{p_k}\right). \tag{A8}$$

This is equal to Eq. (A4). Thus, it is shown that resonant terms which are proportional to t in $\phi_j^{(0)}\partial_{z_j}g_i-g_j\partial_{z_j}\phi_i^{(0)}$ for all j and i are equal to zero. That is to say, all of $\{E^i_{p_1p_2\cdots p_n}\}$ in Eq. (25) are equal to zero.

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